2014 **World Mathematics Team Championship**
Advanced Level
Team Round · Problems

1. Define function

$$F(x,y)=|x-1|+|x-2|+|x-3|+|x-4|+|y-1|+|y-2|+|y-3|.$$

Find the minimum value for $F(x, y)$.

2. Suppose $\boldsymbol{A}=\{(x,y)\,|\,y\geqslant 2x^2\}$ and $\boldsymbol{B}=\{(x,y)\,|\,x^2+(y-a)^2\leqslant 5\}$. If $\boldsymbol{A}\cap\boldsymbol{B}=\boldsymbol{B}$, find the range of values for a.

3. Suppose positive integers a and b are relatively primes and when b is divided by a, 4 and 7 are their remainder and quotient, respectively. Let a_1, a_2, a_3, a_4, $\cdots$ be all the numbers a (in ascending order) that satisfy the above conditions, find a_{2014}.

4. If the range of values for the function $f(x)=\log_{10}\left[\frac{1}{2}x^2-(a+2)x+a^2+4\right]$ is all real numbers, find the domain of $f(x)$.

5. Consider a function $f(x)$ on real numbers $\mathbf{R}$ and satisfies the following conditions:

(a) $f(2+x)=f(2-x)$,

(b) $f(4-x)=-f(4+x)$, and

(c) $f(x)=x^2$ when $0\leqslant x\leqslant 2$.

Find the value for $f(2015)$.

6. Solve the equation $27^{3x^2+2y}+27^{3y^2+2z}+27^{3z^2+2x}=1$ $(x,y,z\in\mathbf{R})$ for (x, y, z).

7. Suppose $f(x)=|x^3-x|-|x^3+x|$. If the equation $f^2(x)+2|f(x)|+n-1=0$ $(n\in\mathbf{R})$ has exactly 3 distinct real roots, find the value for n.

8. Find all possible positive integer solutions x and y for $2\sqrt{x+y}-\sqrt{x}-\sqrt{y}=3$.

9. Let $\{a_n\}$ be a geometric (equal proportion) sequence with $a_n>0$. Suppose $a_4a_{2n-4}=4^n$ $(n\geqslant 3)$ and let S_n be the sum of the first n terms of the sequence $\{\log_2 a_{2n-1}\}$. Find the largest positive integer n that satisfies $S_{2n-1}\leqslant 2015$.

10. Let $P-ABC$ be a tetrahedron that is inscribed inside sphere O. If $AC=BC=6$, $\angle ACB=90^\circ$, and $PB=12$ is the diameter of sphere O, find the volume of $P-ABC$.

11. Suppose, for any positive integers m and n, function f satisfies the following conditions:

(a) $f(1, 1)=2$,

(b) $f(m,n+1)=f(m,n)+(-1)^n\cdot 2$,

(c) $f(m+1,1)=(-1)^m\cdot 2f(m,1)$.

Find the value for $f(2015, 2016)$.

12. If $a,b\geqslant 0$ and $a+b=4$, find the range of values for $(a^2+2)(b^2+2)$.

13. As shown in the Fig. 1, all the centers of semi - circles O_1, O_2, O_3, $\cdots$, and O_n are on AC and each semi - circle tangents to its neighbors at points B_1, B_2, B_3, $\cdots$, and B_n. Suppose CM_1 is also tangent to these semi - circles with points of tangency at M_1, M_2, M_3, $\cdots$, and M_n. If $AB_1=2$ and $\angle M_1O_1C=\theta$, find the value of $O_1B_1+O_2B_2+O_3B_3+\cdots+O_nB_n$ (in terms of θ) when $n\to+\infty$.

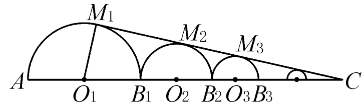

Fig. 1

14. If $2\leqslant x^2y\leqslant 4$ and $-2\leqslant\dfrac{3y^2}{x}\leqslant -1$ where $x, y\in\mathbf{R}$, find the sum of the maximum and minimum values of $\dfrac{y^4}{x^7}$.

15. Let $\triangle ABC_1$ and $\triangle ABC_2$ be isosceles right triangles both with equal side length of 1. If folding along AB to make the two half planes in forming a dihedral angle of $60°$, what is the maximum possible length for C_1C_2?

16. Suppose the three different edge lengths of a rectangular box are m, n, and 1. If m and n satisfy $3m+2n+6mn=9m^2+4n^2+1$, find the length of this rectangular box's main diagonal.

17. Suppose $f(x)=2\cos^2 x-2\sqrt{3}\sin x\cos x$. If the range of values of $f(x)$ is $[0, 1]$ when $x\in\left[\dfrac{\pi}{12}, t\right]$, find the value for t.

18. Suppose the sequence $\{a_n\}$ satisfies $a_1=0$ and $a_{n+1}=a_n+(n+1)\cdot 2^{n-1}$. Find the maximum value among all $C_n=\dfrac{a_n}{3^{n-1}}$.

19. Given a pyramid $P-ABCD$ where base $ABCD$ is a right angle trapezoid, $\angle A=90°$, $AB /\!/ CD$, and $PD\perp$ plane $ABCD$. If $PD=CD=2$ and $AB=AD=1$, find the volume of the sphere that is determined by the 4 points P, B, C, and D.

20. Given positive integers a, b, and c such that $a<c$ and $a+c=2b$. Find the number of 3 - digit numbers $\overline{abc}$ that satisfy these conditions.

Team Round Answers

1. 6.
2. $a\geqslant 10\dfrac{1}{8}$.
3. 28221.
4. $\{x \mid x\neq 4\}$.
5. -1.
6. $(x, y, z)=\left(-\dfrac{1}{3}, -\dfrac{1}{3}, -\dfrac{1}{3}\right)$.
7. 1.
8. $(x, y)=(16, 9)$ or $(9, 16)$.
9. 22.
10. $36\sqrt{2}$.
11. $-2^{2015}-2$.
12. $[32, 36]$.
13. $\dfrac{1+\cos\theta}{2\cos\theta}$.
14. $-\dfrac{11}{144}$.
15. $\sqrt{2}$.
16. $\dfrac{7}{6}$.
17. $\dfrac{\pi}{6}$.
18. $\dfrac{8}{9}$.
19. $\dfrac{8\sqrt{2}}{3}\pi$.
20. 16.

Relay Round · Problems

First Round

1A. Find the number of 3 - digit numbers $\overline{abc}$ (a, b, and c are distinct) such that $|a-c|=5$.

1B. Let $T=$ TNYWR (The Number You Will Receive) and $S=\frac{T}{3}$. If $x+y+z=S$, find the maximum value for $xyz+3(xy+yz+zx)$.

Second Round

2A. Suppose $\overline{abcdefabc}$ is a 9 - digit number with $\overline{def}=2\overline{abc}$. If $\overline{abcdefabc}$ is a product of the squares of 4 distinct prime numbers, find the sum of all possible 3 - digit numbers $\overline{abc}$.

2B. Let $T=$ TNYWR (The Number You Will Receive). Suppose c is the units digit for T. How many 5 - digit numbers in the form of $\overline{12abc}$ can have 3 as remainder when it is divided by 7?

Third Round

3A. Given a sequence $\{a_n\}$ where $a_n=n(n!)$. Let S_n be the sum of the first n terms of $\{a_n\}$. Find the smallest number n so that $S_n \geqslant 2014$.

(Note: $n!=n\times(n-1)\times(n-2)\times\cdots\times2\times1$)

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3B. Let $T=$ TNYWR (The Number You Will Receive). Find the number of negative value solutions to the equation $-x^5+x^4-3x^3+5x^2-2x+T=0$.

Relay Round Answers

First Round	Second Round	Third Round
1A. 72.	2A. 650.	3A. 6.
1B. 1088.	2B. 14.	3B. 0.

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Individual Round · Problems

First Round

1. If $ABCD-A'B'C'D'$ is a rectangular solid, how many tetrahedrons can be formed using the center of this rectangular solid as vertex and three points from vertices A, B, D, B', C', and D' to form the base?

2. If $x^5+5x^4+10x^3+10x^2-5x+1=0(x\neq-1)$, find the value for $(x+1)^4$.

3. Suppose $x\in\mathbf{R}$ and $[x]$ represents the largest integer not larger than x. If the function $f(x)=\frac{[x]}{x}-a$ $(x>0)$ has exactly three zeros, find the range of possible values for a.

4. Let $A(2, 2)$ be a point on the xy-coordinate system, B a point on the line $y=x+1$ and C a point on the x-axis. Find the minimum perimeter for all such possible triangles $\triangle ABC$.

Second Round

5. Suppose, except for PA, the length of all edges in tetrahedron $P-ABC$ has a length of 1. Find the range of possible values for the length of PA?

6. Given that $f(\log_{10}(\log_3 10))=5$ for function $f(x)=\frac{ax^3+b\sin x}{x^2+c}(a, b, c\in\mathbf{R})$. Find the value for $f(\log_{10}(\log_{10}3))$.

7. Suppose sets $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined to be $\boldsymbol{A}=\{x, xy, \log_{10}(xy)\}$ and $\boldsymbol{B}=\{0, |x|, y\}$. If $\boldsymbol{A}=\boldsymbol{B}$, find $\left(x+\frac{1}{y}\right)+\left(x^2+\frac{1}{y^2}\right)+\left(x^3+\frac{1}{y^3}\right)+\cdots+\left(x^{2014}+\frac{1}{y^{2014}}\right)$.

8. Randomly pick two diagonals (including both face diagonals and body diagonals) from a cube. What is the probability that these two diagonals are perpendicular?

Third Round

9. Given $x^5+5\sin x+2m=0$ and $16y^5+5\sin y\cos y-m=0$ where $m\in\mathbf{R}$ and $x, y\in\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$. Find the value for $\cos(x+2y)$.

10. Given a rectangular box $ABCD-A_1B_1C_1D_1$ with the base $ABCD$ being a square and that E and F are points on edges BB_1 and DD_1, respectively, so that $AB=DF=B_1E=2$ and $BE=1$. Find the volume of the pyramid D_1-AEC_1F.

11. Let A, B, and C be the interior angles of triangle $\triangle ABC$ that are opposite to the sides a, b, and c, respectively. If the area of $\triangle ABC$ is $S=\frac{1}{2}bc\cos A=2\sqrt{2}$ and $a=2\sqrt{5-2\sqrt{2}}$, find the value for $b+c$.

12. Suppose $x_i(i=1,2,3,4,5)$ are non-negative real numbers and $x_1+x_2+x_3+x_4+x_5=1$. Use $\max\{x, y\}$ to denote the larger of x and y. Find the smallest possible value for $\max\{x_1+x_2, x_2+x_3, x_3+x_4, x_4+x_5\}$.

Fourth Round

13. Circle C, x-axis, y-axis, and the curve $y=\frac{3}{x}(x>0)$ are tangent to each other as shown in the Fig. 1. Find the radius of circle C.

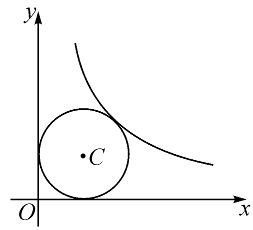

Fig. 1

14. Find the area of the region that is bounded by the curve $|x-2|-|y+1|=|2x-7|$.

Fifth Round

15. As shown in the Fig. 2, A_1, A_2, A_3, and A_4 are points on the x-axis and B_1, B_2, B_3, and B_4 are points on the curve $y^2=kx(k>0)$. Suppose points C_1, C_2, and C_3 are points on A_2B_2, A_3B_3, and A_4B_4, respectively, so that $A_1B_1C_1A_2$, $A_2B_2C_2A_3$, and $A_3B_3C_3A_4$ are all squares with areas, respectively, S_1, S_2, and S_3. If $OA_1=1$ and $S_2=2S_1$, find S_3. (The result cannot contain k).

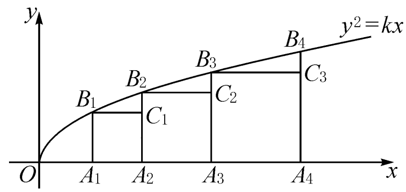

Fig. 2

16. Consider the sequence $\{a_n\}$ with $a_1=\frac{1}{4}$. Denote the sum of its first n terms as S_n. If a_n is the arithmetic average of $\sqrt{S_n}$ and $\sqrt{S_{n-1}}$ for all $n\geqslant 2$, find the value for a_{2014}.

Individual Round Answers

First Round

1. 8.
2. 10.
3. $\left(\frac{3}{4}, \frac{4}{5}\right]$.
4. $\sqrt{26}$.

Second Round

5. $(0, \sqrt{3})$.
6. -5 .
7. 0.
8. $\frac{3}{10}$.

Third Round

9. 1.
10. $\frac{4}{3}$.
11. 6.
12. $\frac{1}{3}$.

Fourth Round

13. $2\sqrt{3}-\sqrt{6}$.
14. 3.

Fifth Round

15. $2+\sqrt{2}$.
16. $1006\frac{3}{4}$.