2013 **World Mathematics Team Championship**
Intermediate Level
Team Round · Problems

1. The following formula can be used to calculate the day of the week:

$$S=(x-1)+\left[\frac{x-1}{4}\right]-\left[\frac{x-1}{100}\right]+\left[\frac{x-1}{400}\right]+y$$

where x = year of the century, y = number of days from Jan. 1 of this year. $[x]$ represents the largest integer which is no more than x. When S is divided by 7, the remainder represents the day of the week (0 = Sunday, 1 = Monday, ⋯, 6 = Saturday). What day of the week is November 25, 2011? (Use numbers from 0 to 6)

2. If $t=\frac{\sqrt{5}-1}{4}$, find $16t^5-20t^3+5t$.

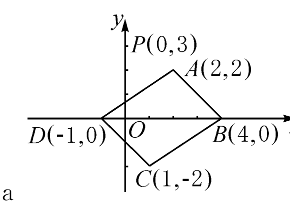

Fig. 1

3. Compute: $\frac{4^{\frac{1}{2013}}}{4^{\frac{1}{2013}}+2}+\frac{4^{\frac{2}{2013}}}{4^{\frac{2}{2013}}+2}+\cdots+\frac{4^{\frac{2012}{2013}}}{4^{\frac{2012}{2013}}+2}$.

4. Given a point $P(0,3)$ and four points A, B, C, and D that form a quadrilateral as shown in the Fig. 1. If point P rotates around point A 180° and becomes P_1, then point P_1 rotates around point B 180° and becomes P_2, then point P_2 rotates around point C 180° and becomes P_3, point P_3 rotates around point D 180° and becomes P_4, point P_4 rotates around point A 180° and becomes P_5, ⋯. If points rotate according to this pattern, what are the coordinates for point P_{2013}?

5. How many ordered pairs of integers (a,b) so that the system of inequalities $\begin{cases}3x-a\geqslant 0\\5x-b<0\end{cases}$, has only two integer solutions $x=1$ and $x=2$?

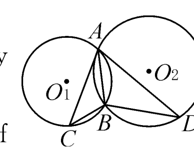

Fig. 2

6. How many triangles with different perimeters can be formed by connecting any three vertices of a $2\times3\times4$ rectangular box?

7. Consider the Fig. 2, $\odot O_1$ and $\odot O_2$ intersect at points A and B. Chord AC of $\odot O_1$ tangents $\odot O_2$ at point A and chord AD of $\odot O_2$ tangents $\odot O_1$ also at point A. If the ratio of areas of $\triangle ABC$ and $\triangle ABD$ is $3:4$, find the ratio of radius of $\odot O_1$ and radius of $\odot O_2$.

8. As shown in the Fig. 3, O is the origin, A is a point on the curve of $y=\frac{1}{x}(x>0)$, and B is a point on curve $y=-\frac{4}{x}(x<0)$. Find the smallest possible area of $\triangle AOB$.

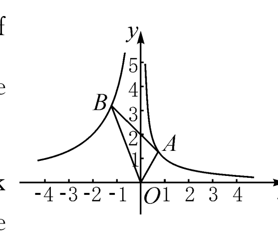

Fig. 3

9. Below is a table which listed numbers from 1 to 10000 as shown. Pick any number from this table and cross off all the numbers on the same row and same column as this selected number. From the remaining uncrossed numbers, pick another number and also cross off the numbers that are on the same row and same column of this number. Continue this way until the 100th numbers was selected. Find the sum of these selected numbers.

1	2	3	…	99	100
101	102	103	…	199	200
201	202	203	…	299	300
…	…	…	…	…	…
9801	9802	9803	…	9899	9900
9901	9902	9903	…	9999	10000

10. As shown in the Fig. 4, both $\triangle OAB$ and $\triangle BCD$ are equilateral triangles with points $A(\sqrt{3},3)$ and C on curve $y=\frac{k}{x}(x>0)$ and points B and D are on the x-axis. Find the coordinates of point D.

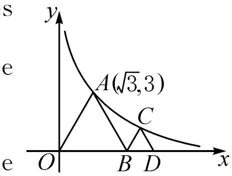

Fig. 4

11. How many 6-digit numbers so that each digit is a prime number and the sum of the 6 digits is 23?

12. Place two squares A and B inside 2 identical fans as shown in the Fig. 5. If the central angle of this fan is 60°, find the ratio of area of B to the area of A.

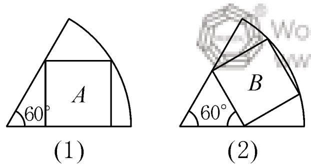

Fig. 5

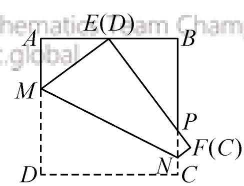

Fig. 6

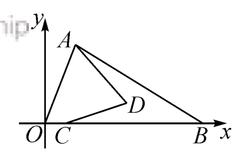

Fig. 7

13. As in the Fig. 6, $ABCD$ is a square of length 4. Fold the bottom edge of this square so that vertex D goes to E which is the midpoint of AB. Let MN be the crease mark of this fold and vertex C is folded to F. If EF intersects BC at P, find the length of PN.

14. Consider the Fig. 7. Let O be the origin. Suppose AOB is a triangle with B on the x - axis and the broken line ADC divides this triangle's area into two halves. If $A=(3,8)$, $D=(8,2)$, and $C=(2,0)$, find the coordinates of B.

15. Find the value of $a^4+b^4+c^4$ if $a+b+c=0$ and $a^2+b^2+c^2=3$.

16. Suppose the parabola $y=ax^2+bx+c$ intersects the x-axis at $A(x_1,0)$ and $B(x_2,0)$ where a,b, and c are all positive integers. If $|x_1|$ and $|x_2|$ are both greater than 1, find the smallest possible value for $a\cdot b\cdot c$.

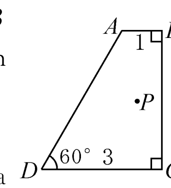

Fig. 8

17. Given a trapezoid $ABCD$ as shown in the Fig. 8 where $\angle B=\angle C=90^\circ$, $\angle D=60^\circ$, $AB=1$, and $DC=3$. What is the probability of randomly selecting a point P inside the trapezoid so that $PA>AB$ and $PD>DC$? (Use $\pi=3$).

18. Given a trapezoid $ABCD$ as shown in the Fig. 9 where $DA \parallel CB$, $\angle A=90^\circ$, BD bisects $\angle ABC$ and CE bisects $\angle DCB$ with E on AB. If $AE:EB=1:7$ and $BC=7$, find the area of trapezoid $ABCD$.

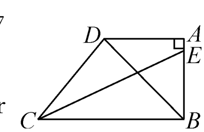

Fig. 9

19. Given a circle $\odot O$ with radius 7 and a chord AB of length 10. If another chord MN of $\odot O$ of length 4 is moving around $\odot O$ and forms a quadrilateral $AMNB$ with AB, find the largest area of all such quadrilaterals.

20. How many different trapezoids can be formed by using 4 segments from a set of 6 segments 1, 2, 3, 4, 5, and 6? (Two congruent trapezoids are considered one trapezoid).

Team Round Answers

1. 5.
2. 1.
3. 1006.
4. (4，1).
5. 15.
6. 7.
7. $\frac{\sqrt{3}}{2}$.
8. 2.
9. 500050.
10. $(2\sqrt{6},0)$.
11. 300.
12. $\frac{8-3\sqrt{3}}{3}$.
13. $\frac{5}{6}$.
14. $\left(\frac{31}{2},0\right)$.
15. $\frac{9}{2}$.
16. 25.
17. $\frac{24-11\sqrt{3}}{24}$.
18. 22.
19. $21\sqrt{5}+14\sqrt{6}$.
20. 28.

Relay Round · Problems

First Round

1A. Andrew and Benjamin start at the same time from Location A walking to Location B. Andrew ′s speed is 10% faster than Benjamin′s speed so Andrew arrives Location B 10 minutes earlier than Benjamin. How many minutes would it take Benjamin to walk from A to B?

1B. *Let* T=TNYWR (The Number You Will Receive).

If $x=a_n$ and $x=b_n$ (n is a natural number greater than 1) be the roots for 2^{nd} degree equation $x^2+3nx+2n^2+1$, find the value for

$$\frac{1}{(a_2-1)(b_2-1)}+\frac{1}{(a_3-1)(b_3-1)}+\cdots+\frac{1}{(a_T-1)(b_T-1)}.$$

Second Round

2A. How many digits does the product $5^{39}\times 4^{22}$ have?

2B. *Let* T=TNYWR (The Number You Will Receive).

Suppose the lengths of all three sides of $\triangle ABC$ are integers less than T. As shown in the Fig. 1, if CD is the height from base AB, how many such triangles are there so that $AC \cdot BC=AB \cdot CD$ and $AB+AC=2BC$?

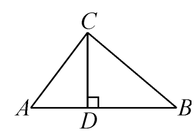

Fig. 1

Third Round

3A. Definition: Lattice Points are points with integer coordinates.

Suppose C is a cirde of radius 2 and center origin. Among all the squares inscribed in circle C, what is the largest number of Lattice Points that can be covered (include the points on the 4 sides of the square)?

3B. *Let* T=TNYWR (The Number You Will Receive).

As shown in the Fig. 2, $\triangle ABD$, $\triangle BCD$, and $\triangle ACD$ have areas of 9, 17, and T, respectively. Find the area of $\triangle DEC$.

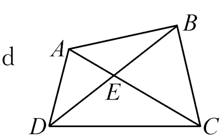

Fig. 2

Relay Round Answers

First Round

1A. 110.

1B. $\frac{109}{444}$.

Second Round

2A. 41.

2B. 8.

Third Round

3A. 13.

3B. 8.5.

Individual Round • Problems

First Round

1. How many different triangles have positive integers as side lengths and perimeter of 15? (Two congruent triangles are considered one triangle).

2. If $2x+3y-2z=0$ and $2x-3y+4z=0$, find the value for

$$\frac{(3x-2y)^2-(3y-5z)^2}{(3x-2y)(3y-5z)}.$$

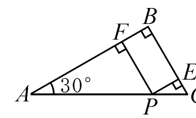

Fig. 1

3. In $\triangle ABC$, $AB=AC=\sqrt{5}+1$ and $\angle B=72°$. Find the length of BC.

4. As in the Fig. 1, in $\triangle ABC$, $\angle A=30°$, $\angle B=90°$, and $AC=20$. Let P be a point on AC such that $PE\perp BC$ at point E and $PF\perp AB$ at point F. Find the area of the largest such rectangle $BEPF$.

Second Round

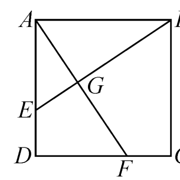

Fig. 2

5. As shown in the Fig. 2, E and F are points on the sides of square $ABCD$ so that $AE=2ED$, $DF=2FC$, and AF and BE intersects at G. Find the proportion $AG:GF$. (Express answer in simplest fraction.)

6. Find all positive integer solutions (x, y) for equation $x^2-y^2-x-5y+6=0$.

7. AS shown in the Fig • 29, there are 8 pieces of wood blocks. Two of them have the letter 'W' on them, two with the letter 'M', two with the letter 'T', and the last two have the letter 'C' on them. If four blocks are randomly picked out from these eight blocks, what is the probability that these four chosen blocks would compose into the word 'WMTC'. (Express answer in simplest fraction.)

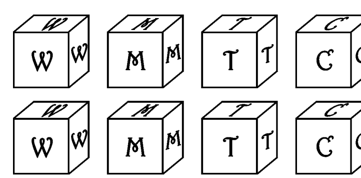

Fig. 3

8. If the equation $|x^2-mx|=1$ with x as variable has three distinct real roots, find the value for m.

Thord Round

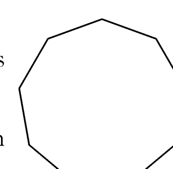
Fig. 4

9. AS shown in the Fig. 4, if each side of a regular nonagon (polygon of 9 sides) is of length 1, find the largest difference in length of two of diagonals.

10. Natural numbers from 1 to 100 are placed into A and B two groups in which the number 30 is in Group A. If the number 30 is moved from Group A to Group B, then the two averages of the numbers in these two groups both increased by 0.5. How many numbers are in Group A after the number 30 is moved to Group B?

11. Suppose two non-zero real numbers x and y satisfy $|x|+y=2$ and $|x|y+x^3=0$. Find the value for y.

12. Given that real numbers x, y, and z satisfy

$$\begin{cases} 2x^3=3y^3=4z^3, & ① \\ \sqrt[3]{2x^2+3y^2+4z^2}=2+\sqrt[3]{12}+\sqrt[3]{16}, & ② \\ xyz>0. & ③ \end{cases}$$

Find the value for $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

Fourth Round

13. In $\triangle ABC$, the lengths of the opposite sides to $\angle A$, $\angle B$, and $\angle C$ are a, b, and c, respectively. If $\frac{a}{b}=\frac{a+b}{a+b+c}$ and $\angle A=30^\circ$, find the degree measure for $\angle B$.

14. Suppose each of these 30 numbers $a_1, a_2, \cdots, a_{30}$ can only take on value of -2 or 0 or 1. If $a_1+a_2+\cdots+a_{30}=-18$ and $(a_1-1)^2+(a_2-1)^2+\cdots+(a_{30}-1)^2=126$, how many of these 30 numbers take on a value of -2?

Fifth Round

15. We call a number that can be written as the sum of the squares of 2 different positive integers a "*Hope Number*".
For examples, 5 and 34 are *Hope Numbers*, since $5=1^2+2^2$, $34=3^2+5^2$.
How many *Hope Numbers* are there from 1 to 100?

16. Suppose a parabola $y=\frac{1}{4}x^2$ and a straight line $y=-\frac{1}{2}x+6$ intersect at points A and B. Find the coordinates of all possible points P on this parabola so that $\triangle ABP$ is a right triangle.

Individual Round Answers

First Round

1. 7.
2. $\frac{33}{28}$.
3. 2.
4. $25\sqrt{3}$.

Second Round

5. $\frac{6}{7}$.
6. (6, 4), (1, 1).
7. $\frac{8}{35}$.
8. ± 2.

Third Round

9. 1.
10. 70.
11. 1.
12. $\frac{1}{2}$.

Fourth Round

13. 60°.
14. 13.

Fifth Round

15. 29.
16. $(-2, 1)$ or $(14, 49)$.