2013 **World Mathematics Team Championship**
Advanced Level
Team Round · Problems

1. Find the range of values for real number m so that $y=2x$ contains points that satisfy the conditions $\begin{cases} x+y-1\leqslant 0 \\ x-2y-1\leqslant 0. \\ x\geqslant m \end{cases}$

2. How many lattice points (x,y) (including those on the edges) satisfy the inequality $|x+2|+|2x-3y+1|\leqslant 6$. (Note: Lattice points are points with integer coordinates.)

3. Suppose the sides a, b, and c of $\triangle ABC$, which correspond to interior angles A, B, and C, respectively, form a geometric (equal proportion) sequence, find the value range of $y=\dfrac{\sin B+\cos B}{1+\sin 2B}$.

4. Given sets $A=\{x\,|\,\ln(x^2+4x+1)=0\}$ and $B=\{x\,|\,x^2+2(a+1)x+a^2-1=0\}$. If $A\cup B=A$, find the range of values for real number a.

5. Arrange 240 small equal sized cubes with length 1 to makes up a $5\times 6\times 8$ rectangular solid. For this rectangular solid, color three adjacent faces that share a common vertex. Now unpack this rectangular solid and pick one small cube at random. Let X be the number of colored faces. Find the expected (average) value EX. (Note: $EX=x_1p_1+x_2p_2+\cdots+x_ip_i+\cdots+x_np_n$).

6. Let $[x]$ be the greatest integer less than or equal to x. How many roots does the equation $x+1=\underbrace{\left[\frac{x}{2}\right]+\left[\frac{x}{3}\right]+\cdots+\left[\frac{x}{100}\right]}_{99[\ ]\text{s}}$ have?

7. Suppose a sequence $\{a_n\}$ satisfies $a_{n+1}=\dfrac{a_n}{1+(3n^2+n-1)a_n}$ for any natural number n and $a_1=1$. Find the value of a_{10}.

8. If the graph of function $y=\log_2\dfrac{x+3}{2x-4}$ is symmetric about the point (a,b), find the value of ab.

9. Let S_n be the sum of the first n terms of the sequence $\{a_n\}$ and $S_n=-2n^2+45n$. Find the maximum value of $\dfrac{S_n}{a_n}$ when $n<20$.

10. Line l passes through point $P(-2,0)$ and intersects circle $x^2+y^2=1$ at two different point A and B. Find the range of values of $|PA|+|PB|$.

11. Let S_n be the sum of the first n terms of the sequence $\{a_n\}$ and for each $n=1,2,3,\cdots$, the equation $x^2-2a_nx-4a_n=0$ has one solution $2S_n-2$. Find the value of a_{100}.

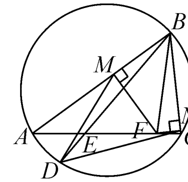

Fig. 1

12. As shown in the Fig. 1, acute triangle $\triangle ABC$ is inscribed in a circle with $AB=10$, $BC=6$, and $\angle ABC=60°$. Suppose both points E and F are on AC and satisfy $\angle ABE=\angle FBC$. Let M and N be points on AB and BC, respectively, so that $FM\perp AB$ and $FN\perp BC$. Extend BE until it intersects the circumscribed circle of $\triangle ABC$ at point D. Find the area of quadrilateral $BNDM$.

13. If the tangent of each angle of $\triangle ABC$ is a positive integer and the shortest side of $\triangle ABC$ has a length of $\sqrt{5}$, find the length of the longest side.

14. If the lengths of three sides of $\triangle ABC$ are consecutive even numbers and $\angle A = 2\angle B$, find the perimeter of $\triangle ABC$.

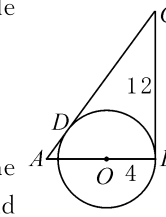

Fig. 2

15. As in the Fig. 2, CD and CB tangent to circle O at point D and B and the extension of line BO intersects the extension of CD at point A. If $OB = 4$ and $BC = 12$, find the length of AD.

16. Suppose a fixed sphere M and a fixed point P are on the same side of a plane α and let N be the point on α that is the projection of P so that $MP = \sqrt{2}$, $PN = \sqrt{2+\sqrt{3}}$, and $\angle MPN = 120°$. If point Q is a point on plane α giving a minimum value to $QM + QP$, find $\angle MQP$.

17. Place 1638 pieces of 1×1 square papers side-by-side without overlap to form a pair of rectangles so that:

(1) They have the same width,

(2) The lengths of their sides are all prime numbers,

(3) The ratio of their areas is between 2 and 3.

Among all possible rectangles that satisfy these conditions, find the length of the side of the rectangle with the largest area.

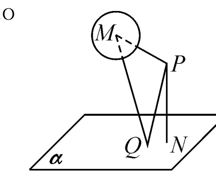

Fig. 3

18. Suppose the equation $m - \frac{1}{2}x = |x^2 - 4x|$ of x has 4 different real roots. Find the range of values for real number m.

19. There are 2013 points on a plane where any 3 points can form a triangle (that means no 3 points are collinear). Divide these points into 67 groups so that each group has at least 3 points. Let m_i be the number of triangles that can be formed in group i. Find the minimum value for $\sum_{i=1}^{67} m^i$.

20. Divide natural numbers from 1 to 100 into two groups with 50 numbers in each group. Let the numbers in the first group be $a_1, a_2, \cdots, a_{50}$ and the numbers in the second group be $b_1, b_2, \cdots, b_{50}$. How many possible values can $|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{50} - b_{50}|$ represent?

Team Round Answers

1. $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

2. 33.

3. $\left[\frac{\sqrt{2}}{2}, 1\right)$.

4. $a \leqslant -1$ or $a = 1$.

5. $\frac{59}{120}$.

6. 0.

7. $\frac{1}{892}$.

8. $\frac{1}{2}$.

9. $\frac{253}{3}$.

10. $(2\sqrt{3}, 4]$.

11. $\frac{1}{10100}$.

12. $15\sqrt{3}$.

13. 3.

14. 30.

15. 3.

16. $30°$.

17. 403.

18. $2 < m < \frac{81}{16}$.

19. 273325.

20. 1226.

Relay Round · Problems

First Round

1A. The three views of a solid are shown in the Fig. 1:

Find the volume of this solid.

1B. Let T = TNYWR (The Number You Will Receive).

If $K=\sqrt{\frac{\mathrm{T}}{5\pi}}$, find the value of $\sum_{a=1}^{k} a^3$.

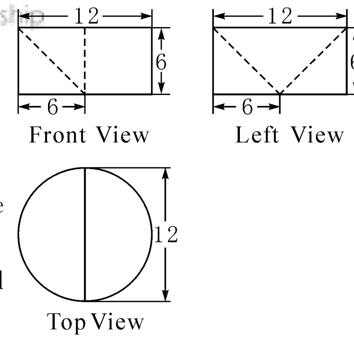

Fig. 1

Second Round

2A. Suppose both circle M of radius R and circle O of radius r are tangent to both sides of a right angle P.

If $R>r$ and circle M passes through the center of circle O, find the value of $\frac{R}{r}$.

2B. Let T = TNYWR (The Number You Will Receive).

Let $K=\mathrm{T}-\sqrt{2}$. If the areas of the three adjacent faces that share a common vertex of a rectangular box are $60K$, $20\sqrt{5}K$, and $24\sqrt{5}K$. Find the volume of the circumscribed sphere of this box. (Use $\pi=3$)

Third Round

3A. How many positive integers n so that $(n+3)$ divides (n^2+7) evenly.

3B. Let T = TNYWR (The Number You Will Receive).

Find the positive integer n that will make

$$(n^3-21n^2+122n-167)(n^2-8n+20)(n^2+\mathrm{T}n-69)$$

a prime number.

Relay Round Answers

First Round

1A. 180π.

1B. 441.

Second Round

2A. $2+\sqrt{2}$.

2B. 972π.

Third Round

3A. 3.

3B. 7.

Individual Round · Problems

First Round

1. Find the minimum value for function $y=x^2+\frac{1}{\sqrt{x}}$.

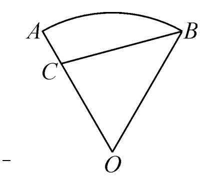

Fig. 1

2. Find the value range of function $f(x)=x^2-20\sqrt{4-x^2}-2$.
3. If the lengths of three sides of $\triangle ABC$ are consecutive even numbers and interior angles A, B, and C satisfy, $\sin A:\sin B:\sin C=2:3:4$, find the area of $\triangle ABC$.
4. Let C be a point on OA which is the radius of sector OAB as shown in the Fig. 1.

   Suppose $\angle ACB=105°$, $AC=4-2\sqrt{3}$, and $BC=3\sqrt{2}-\sqrt{6}$.

   Find the area of sector OAB. (Use $\pi=3$)

Second Round

5. Find the number of positive integer solutions for equation about a, b, c, d, e:

$$a+b+c+d+e=9.$$

6. Define function $f(x)=2x-3[x]$ in interval $[-1,2]$ where $[x]$ represents the greatest integer less than or equal to x. Find all possible integer values for $f(x)$.
7. In rectangular coordinate system xOy, lines $l_1: 4x+3y-22=0$, $l_2: 7x+4y-26=0$, and $l_3: x+2y-8=0$ form a triangle. Find the area of this triangle.
8. If $\cos 5\theta=x\cos^5\theta+y\cos^3\theta+z\cos\theta\ (x, y, z\in\mathbf{R})$ holds for any $\theta\in\mathbf{R}$, find the value of $x^{\frac{z}{y}}$.

Third Round

9. Define function $f(x)=\begin{cases} e^{|\ln x|}, & 0<x\leqslant 3, \\ |x-6|, & x>3. \end{cases}$ Let a, b, c, and d be four distinct positive numbers and $f(a)=f(b)=f(c)=f(d)$. Find the range of values for $abcd$.

10. Define function $f(x)=\dfrac{\sin\frac{\pi x}{3}\sin\pi x}{2+2\cos\pi x}$ in interval $(-1,1)$. If $f\left(2^x-\frac{1}{4}\right)\geqslant\frac{1}{4}$, find x.

11. As in the Fig. 2, $ABC-A_1B_1C_1$ is a prism with $AC=CB$, $AB=4$, $B_1B=2$, $\angle A_1AB=60°$, and plane $AA_1B_1B\perp$ plane ABC. Suppose M is a point moving along A_1B_1. What is the length of A_1M when the planar angle of A_1-BM-C is at minimum?

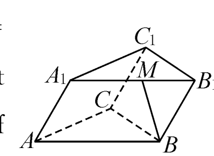

Fig. 2

12. Given an expression $\frac{a_1^2}{a_2}+\frac{a_2^2}{a_3}+\cdots+\frac{a_{2012}^2}{a_{2013}}+\frac{a_{2013}^2}{a_1}$ with a minimum value of 2013 where a_1, a_2, $\cdots$, and a_{2013} are positive real numbers. Find a_{2013}.

Fourth Round

13. If inequality $|x-2|-|x+t|\leqslant 5$ holds for any real number x, find the range of values of real number t.

14. Given functions $f(x)=\log_2 x+x-2$ and $g(x)=2^{1-x}-x-12$.
Define $F(x)=f(x-3)g(x+3)$. If $F(x)$ has all its zeros in the interval (m, n) $(m, n\in \mathbf{Z})$, find the minimum value for $n-m$.

Fifth Round

15. As shown in the Fig. 3, in tetrahedron $P-BCD$, $PB=PC=PD=BC=\sqrt{3}$. If $\angle BCD=90°$ and $CD=1$, find the tangent of the planar angle of B-PD-C.

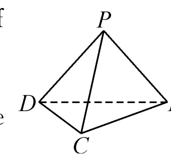

Fig. 3

16. Suppose there are 30 yellow, blue, and white balls, 10 of each. Now put these 30 balls in some way into Bag A and Bag B so that the products of the numbers of balls in each color in each bag are the same. Assume each bag has balls of all three colors. What is the probability of Bag A has no less than 3 yellow balls?

Individual Round Answers

First Round

1. $5\times 2^{-\frac{8}{5}}$.

2. $[-42, 14]$.

3. $3\sqrt{15}$.

4. 2.

Second Round

5. 70.

6. -2, -1, 0, 1, 2.

7. 5.

8. $\frac{1}{2}$.

Third Round

9. $(27, 35)$.

10. $\log_2 3-2\leqslant x<\log_2 5-2$.

11. 3.

12. 1.

Fourth Round

13. $[-7, 3]$.

14. 11.

Fifth Round

15. $\frac{3\sqrt{2}}{2}$.

16. $\frac{21}{25}$.