2012 **World Mathematics Team Championship**
Advanced Level
Team Round · Problems

1. Given a tetrahedron $PABC$ in which $\angle APB = \angle BPC = \angle CPA = 90^\circ$ and $PA = PB = PC = 1$. If $PABC'$s inscribed sphere has a radius of r, then $\frac{1}{r}$ is between which two consecutive integers?

2. If $(5x+2y)^{25} + x^{25} + 6x + 2y = 0$, find the value for $6x+2y$.

3. In Fig. 1, let $ABCD$ be a tetrahedron with edge $AD = \sqrt{2}$ and all the other edges have a length of 1. Suppose M and N are midpoints of AB and CD, respectively. Find the minimum distance moving from M to N over the surface of $ABCD$.

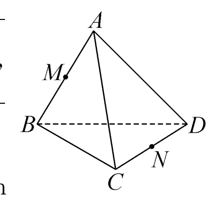

Fig. 1

4. Let x be any real number and let $[x]$ represents the greatest integer less than or equal to x. For example, $[2.3]=2$ and $[-2.3]=-3$. Find all possible real solutions for equation $\left[\frac{x}{2}\right] + \left[\frac{x}{4}\right] + \left[\frac{x}{6}\right] + \left[\frac{x}{8}\right] + \cdots + \left[\frac{x}{2012}\right] = x$.

5. Randomly pick five numbers from ten numbers 1, 2, 3, ⋯, 10. Arrange these 5 numbers from small to large and label them a_1, a_2, a_3, a_4, and a_5. Arrange the remaining five numbers from large to small and label them b_1, b_2, b_3, b_4, and b_5. Find the value for

$$|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5|.$$

6. If the system of equations $\begin{cases} mx^2 + x + 2m - 1 = 0 \\ x^2 + (m-1)x = 0 \end{cases}$ has only one real solution for x, find m.

7. How many functions can be defined so that the function's domain is

$$A = \{ab \mid ab - a - 2b - 2 = 0, a, b \in \mathbf{Z}\}$$

and its range is $B = \left\{x \mid \sin\left(x - \frac{\pi}{6}\right) = 1, 0 < x < 5\pi\right\}$?

8. If $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right)$ and $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$, find the maximum value for $\frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma}$.

9. If integer m satisfies the equation $\left(1 + \frac{1}{m}\right)^{m+1} = \left(1 + \frac{1}{2012}\right)^{2012}$, find the value for m.

10. Suppose n is a natural number larger than 0. If $f(n)$ represents the number of roots in $[0, \pi]$ for equation $\sin x = \cos(nx)$, find $f(1) + f(2) + f(3) + \cdots + f(100)$.

11. Let $\{a_n\}$ be a sequence that satisfies $a_{n+3} - a_{n+2} + a_{n+1} - a_n = 0$. If S_n represents the sum of the first n terms of $\{a_n\}$ and $S_{2012} = 2012$, find $a_1 + a_3$.

12. Suppose the straight line $2x - y - 12 = 0$ and the parabola $y^2 = 4x$ intersect at points A and B. Let C be another point on the parabola. If $\angle ACB = 90^\circ$, find coordinates of all possible C.

13. If real numbers x and y satisfy $\begin{cases} x+3y-23\leqslant 0, \\ 2x+y-11\geqslant 0, \\ x-2y+2\leqslant 0, \end{cases}$ find the range of values of $\dfrac{x+y+4}{x+1}$.

14. Arrange the positive roots for equation $x\cdot\cos x+\sin x=0$ in ascending order $a_1, a_2, \cdots, a_n, \cdots$. Among the following conclusions, select the correct statements.

① $0<a_{n+1}-a_n<\dfrac{\pi}{2}$;

② $\dfrac{\pi}{2}<a_{n+1}-a_n<\pi$;

③$2a_{n+1}>a_{n+2}+a_n$;

④$2a_{n+1}<a_{n+2}+a_n$.

15. Find the intervals where the function $y=\left|\cos\left|\dfrac{\pi}{2}+2x\right|\right|$ is monotonically decreasing.

16. Suppose the graph of quadratic function $f(x)=-x^2+bx+c\ (\Delta=b^2+4ac>0)$ intersects x-axis at A and B. If point $P(x_0, f(x_0))$ where $f(x_0)\neq 0$ is on the curve and PA is perpendicular to PB, find $f(x_0)$.

17. Let $A=(1,1)$. Suppose B and C are two points on the parabola $y=x^2$ such that AB is perpendicular to BC. Among all possible B and C, find the area of the smallest circumscribed circle of $\triangle ABC$.

18. If the solution set for the inequality $\sqrt{x+1}+2x\leqslant b$ is $[-1,3]$, find the solution set for the inequality $||x-1|-10|\leqslant b$.

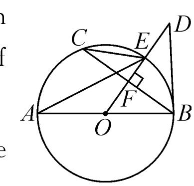

Fig. 2

19. As in Fig. 2, AB is the diameter of the Circle O, $OD\perp BC$ at F on BC and intersects the circle at E. If $\angle AEC=\angle ODB$, $OA=5$, and $BC=8$, find DE.

20. Find all closed intervals M so that the function $f(x)=\dfrac{1}{3}x^2+\dfrac{2}{3}x-\dfrac{11}{3}$ has M as both its domain and range.

Team Round Answers

1. 4 and 5.
2. 0.
3. 1.
4. 0, 8, 10.
5. 25.
6. $\dfrac{1}{2}$ or 0.
7. 36.
8. $\dfrac{\sqrt{2}}{2}$.
9. -2013.
10. 5025.
11. 2.
12. $(1,2)$ or $(4,-4)$.
13. $\left[\dfrac{17}{9}, \dfrac{13}{3}\right]$.
14. ② and ④.
15. $\left[\dfrac{k\pi}{2}+\dfrac{\pi}{4}, \dfrac{k\pi}{2}+\dfrac{\pi}{2}\right], k\in\mathbf{Z}$.
16. 1.
17. π.
18. $[-17,-1]\cup[3,19]$.
19. $\dfrac{10}{3}$.
20. $[-4,-1]$ *or* $\left[-4, \dfrac{1+3\sqrt{5}}{2}\right]$.

Relay Round · Problems

First Round

1A.If $f(\cos\alpha)=\cos(2012\alpha)$, find the value for $f\left(\sin\dfrac{\pi}{2012}\right)$.

1B.Let T = TNYWR (The Number You Will Receive).

Let $f(x)$ be an odd function defined on $\mathbf{R}$ and $f(x+2)=-f(x)$ for all real x. If $f(3)=\dfrac{2m-3}{m+1}$ and $f(1)>-T$, find the range of values for m.

Second Round

2A.Given that $\log_m x$, $\log_n x$, and $\log_2 x$ form an arithmetic (equal difference) sequence and $x\neq 1$. Find the value for $\log_m n \cdot \log_2(2m)$.

2B.Let T= TNYWR (The Number You Will Receive).

Given that $\{a_n\}$ is an arithmetic(equal difference) sequence. If S_n is the sum of the first n terms and $S_T=S_{T+11}$, find a_8.

Third Round

3A.If $f(x)=\dfrac{1}{2^a+x}+\dfrac{1}{2^a-x}-1$ and $f(1)+f(-1)=\dfrac{2}{3}$, find a.

3B.Let T = TNYWR (The Number You Will Receive).

Suppose the length of the square $ABCD$ is T. Let E be a point that moves along the circle with AD as the diameter. Find the range of values for $EB+EC$.

Relay Round Answers

First Round

1A.-1.

1B.$-1<m<\dfrac{2}{3}$.

Second Round

2A.2.

2B.0.

Third Round

3A.1.

3B.$[\sqrt{2},\sqrt{10}]$.

Individual Round・Problems

First Round

1. If $M=\cos^2 x+x\sin x$ where $0<x<\frac{\pi}{2}$, find the integer portion of M.

2. If all the neighboring intersecting points of straight line $y=1$ and curve $f(x)=\sin\left(\omega x+\frac{\pi}{3}\right)$ have a fixed distance of 4, find $f(1)+f(2)+\cdots+f(2013)$.

3. Given a geometric (equal proportion) sequence $\{a_n\}$ with both its first term $a_1>0$ and common ratio $q>0$. Let $b_n=\sqrt{a_n}+\sqrt{a_{2012-n}}$ where n is a natural number and $0<n<2012$. If b_k is the smallest term of the sequence $\{b_n\}$, find k.

4. Given that the straight line $y=m(m<0)$ and the curve $y=\cos x$ intersect on the right side of the y-axis. Arrange the horizontal coordinates of these interceptions in ascending order $x_1, x_2, x_3, \cdots$. If x_1, x_2, and x_3 form a geometric (equal proportion) sequence, find m.

Second Round

5. Let $f(x)=x^3-6x^2+13x-6$ and $f(a)=7, f(b)=1$, find $a+b$.

6. If $S=\min\left\{\frac{3}{|x-1|}, \frac{1}{|x-9|}\right\}$ for real numbers $x\neq 1$ or 9, find $S_{\max}$.

7. Find the range of values of slope k so that the straight line $y=kx+1$ would intersect the ellipse $3x^2+y^2=1$ at the First Quadrant ($x>0$ and $y>0$).

8. Given a sequence $\{a_n\}$ with an explicit formula $a_n=(-1)^n(2n-1)$ where n is positive integers. Let S_n be the sum of this sequence's first n terms. Write the explicit formula for S_{2k}.

Third Round

9. Find the acute angle x satisfying the equation $(\sin 2x+\cos x)(\sin x-\cos x)=2\cos^2 x$.

10. Given a right triangle with hypotenuse of length $\sqrt[4]{x-2}$. If one of the other sides has a length of $\sqrt[4]{15-3x}$, find the value range of the length of its third side.

11. Given non-negative numbers a, b, c, x, y, and z with $a+b+c+x+y+z=1$, and $abc+xyz=\frac{1}{36}$, find the largest possible value for $abz+bcx+cay$.

12. Let h_a, h_b, and h_c be the heights of BC, CA, and AB, respectively, in $\triangle ABC$. If $h_a : h_b : h_c=6:4:3$, find $\tan(C)$.

Fourth Round

13. Let O be the vertex of the parabola $y^2=2px(p\neq 0)$ and let A and B be two non-vertex points on this parabola. If the slopes of OA and OB are k_1 and k_2, respectively, and k_1 and k_2 are also the roots of equation $x^2+4x-2=0$, find the slope of straight line AB.

14. Given $\triangle ABC$. Let a, b, and c be the opposite sides to interior angles A, B, and C, respectively. If $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$, find the maximum possible value for $\sin(A)$.

Fifth Round

15. Suppose $y=f(x)$ is an even function and $[f(x_1)-f(x_2)](x_1-x_2)<0$ for any $x_1, x_2\in(-\infty,0]$. Find the range of values for x that satisfy $f(x+1)<f(2x-1)$.

16. Consider a cube $ABCD\text{-}A_1B_1C_1D_1$ of edge length 1. Let O_1 be the sphere tangents to each of this cube's 12 edges and let sphere O_2 be the sphere tangents to each of this cube's 6 faces. Suppose M_1 and M_2 represent the area of the regions that are intersected by the plane D_1AC with spheres O_1 and O_2 respectively. Find M_1-M_2.

Individual Round Answers

First Round

1. 1.

2. $\frac{1}{2}$ or $-\frac{1}{2}$.

3. 1006.

4. $-\frac{1}{2}$.

Second Round

5. 4.

6. $\frac{1}{2}$.

7. $(-\sqrt{3},0)$.

8. $S_{2k}=2k$.

Third Round

9. $\frac{\pi}{3}$.

10. $(0,\sqrt[4]{3})$.

11. $\frac{1}{108}$.

12. $-\sqrt{15}$.

Fourth Round

13. $\frac{1}{2}$.

14. $\frac{\sqrt{15}}{8}$.

Fifth Round

15. $(-\infty,0)\cup(2,+\infty)$.

16. $\frac{\pi}{4}$.