2011 World Mathematics Team Championship Intermediate Level

Team Round · Problems

1. Given that a, b, c are different prime numbers and m, n, p are natural numbers not less than 1. If a natural number $N=a^m b^n c^p$, find the number of possible factors of N.

2. (a) Convert 3231 in base 4 to a base 7 number.

(b) Find the sum of digits of this base 7 number. (*Perform the summation in base 7 but express the answer in decimal form.*)

3. The equation $x^2+(a-1)x+ab-2=0$ in terms of x has two distinct real roots for any real number a. Find the range of values for real number b.

4. What is the smallest number of circular papers of radius 1 needed to cover a large circle of radius 2?

5. If the sum of n ($n>1$) consecutive positive integers is 10, find the first term of all such possible sequences.

6. 100! means the product of all the positive integers less than or equal to 100. What is the highest power of 2 that divides this number?

7. Suppose the decimal part of $\sqrt{2}$ is N. Then

$$\begin{aligned}N &= \sqrt{2}-1 \\ &= \frac{1}{\sqrt{2}+1} \\ &= \frac{1}{2+(\sqrt{2}-1)} \\ &= \frac{1}{2+\frac{1}{\sqrt{2}+1}} \\ &= \frac{1}{2+\frac{1}{2+(\sqrt{2}-1)}} \\ &= \frac{1}{2+\frac{1}{2+\frac{1}{\sqrt{2}+1}}} \\ &= \cdots \\ &= \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}.\end{aligned}$$

This is the continued fraction representation of the decimal part of $\sqrt{2}$. Let N' be the decimal part of $\sqrt{5}$. Follow the pattern for N, find the continued fraction representation for N'.

8. Suppose we have ten wooden sticks of lengths 1, 2, 3, 4, …, 10. Select several of them and place one end against another in some manner to form a square. If there are n different kinds of squares that are configured this way, find n. (*Squares that are configured using the same number of wooden sticks and wooden sticks of same length, even if the placement orders are different, are considered the same square and counted only once.*)

9. Given an equation $x^4-4x=1$. If $s=$ sum of all its real roots and $p=$ product of all its real roots, find $s+p$.

10. Find a prime number p so that $109p+4$ is a perfect square.

11. Given a circle O. Suppose A is the largest regular n-polygon that can fit inside Circle O and B is the smallest regular n-polygon that can fit outside Circle O. Find, in terms of n, the ratio of the length of an edge of Polygon A to the length of an edge of Polygon B.

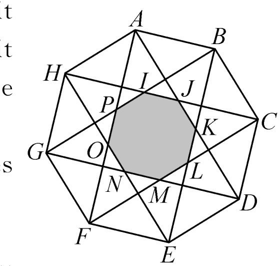

Fig. 1

12. If n and k are positive integers and there is only one k that satisfies $\frac{8}{15}<\frac{n}{n+k}<\frac{7}{13}$, find the largest value for n.

13. If the interior angles of a convex n-polygon $A_1A_2\cdots A_n$ $(n>4)$ are all integral multiples of $15°$ and $\angle A_1+\angle A_2+\angle A_3=285°$, find the largest value for n.

14. As in the Fig. 1, given the regular octagon $ABCDEFGH$ has edge length of 1. Find the area of the shaded regular octagon $IJKLMNOP$.

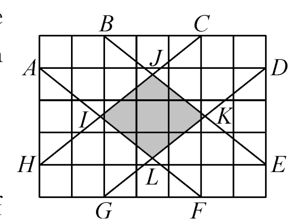

Fig. 2

15. If x, y, and t are real numbers that satisfy the equation $x+y=x^3+y^3=x^5+y^5=t$, list all possible values for t.

16. Fig. 2 is a 7×5 grid that consists of 35 1×1 squares. Find the area of the shaded diamond $JKLI$.

17. Given a circle with center at O and radius 3 as in the Fig. 3. Fit two circles, circle O_1 of radius 1 and circle O_2 of radius 2, externally tangent to each other and internally tangent to circle O. Find the radius of circle C (which is not drawn in the figure 3) that internally tangent to circle O and externally tangent to circles O_1 and O_2.

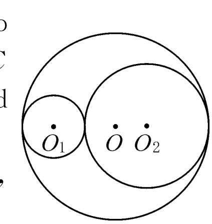

Fig. 3

18. Given that a, b, and c are distinct integers. If $abc<0$ and $a^2+b^2+c^2=529$, find the smallest and largest values for $a+b+c$.

19. If x, y, a, c are all nonzero real numbers and satisfy $x+y=a$, $\frac{x}{y}=c+1$, $x^2-y^2=c^2-1$, and $a^2=c^2+c+1$, find c.

20. As in the Fig. 4, AB and CD are diameters of $\odot O$ and perpendicular to each other. $\odot P$ is tangent to OA at G and tangent to OD at F and tangent to $\odot O$ at E. Point H is on $\odot P$'s $\overset{\frown}{GE}$. If $\odot P$'s $\overset{\frown}{HE}$ has the same length as $\odot O$'s $\overset{\frown}{AE}$, find $\angle HPG$.

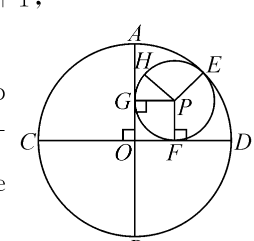

Fig. 4

Team Round Answers

1. $(m+1)(n+1)(p+1)$.
2. (a) 456_7; (b) 21_7.
3. $-2<b<1$.
4. 7.
5. 1.
6. 97.
7. $\dfrac{1}{4+\dfrac{1}{4+\dfrac{1}{4+\cdots}}}$.
8. 23.
9. 1.
10. 113.
11. $\cos\dfrac{180^\circ}{n}$.
12. 112.
13. 10.
14. $\sqrt{2}-2$.
15. -2, -1, 0, 1, 2.
16. $\dfrac{169}{40}$.
17. $\dfrac{6}{7}$.
18. -37, 29.
19. $1\pm\sqrt{3}$.
20. $(2-\sqrt{2})\cdot 45^\circ$.

Relay Round · Problems

First Round

1A. Given

$$x(x+2y-2)=10,$$
$$y(y+2z-2)=12,$$
$$z(z+2x-2)=13.$$

Find the smallest among all the possible values as averages for x, y, and z.

1B. Let T＝TNYWR (The Number You Will Receive).

Suppose you have a large piece of white paper on your desk. Place a circular cardboard of diameter $6|T|$cm on this white paper. As in the Fig. 10, move this cardboard following its diameter MG to the right by $\frac{3}{2}|T|$cm. Find the area that is swept by the right half $\overset{\frown}{CGD}$ of this circle because of this move.

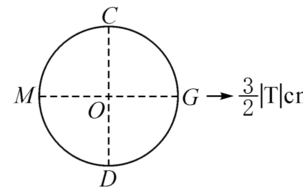

Fig. 10

Second Round

2A. A and B leave towns M and N, respectively at the same time and drives at a constant speed back and forth between town M and town N. B drives at constant 40km per hour on the same highway as A. A arrives town N 20 minutes after first passing B, whereas B arrives town M 45 minutes after passing A. How many minutes will A and B pass each other for the 34^{th} time after they first started?

2B. Let T＝TNYWR.

Suppose there are T number of people. Their skin color is either white or light black and their eye pupil color is either blue or brown. If there are 860 persons who have blue eye pupil color and white skin color, 980 persons have light black skin color, and 520 persons have brown eye pupil color, then how many among these T number of people have brown eye pupil color and light black skin color?

Third Round

3A. Suppose a, a, and b are the lengths of an isosceles triangle and b, b, and a are the lengths of another isosceles triangle with $a>b$. If the vertex angles of these two triangles are supplementary to each other, find $\frac{a^2+b^2}{a^2-b^2}$.

3B. Let T＝TNYWR.

As in the Fig. 11, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, $\triangle A_3B_3C_3$, and $\triangle A_4B_4C_4$ are equilateral triangles. These four triangle are located on the same side of line l, and they all have one side aligned on line l. Points D_1, D_2, D_3 are the side trisectors of the three larger triangles and they are the side bisectors of the three smaller triangles. If the area of $\triangle A_1B_1C_1$ is T^{10}, find the area of the region that is enclosed by the broken line $B_1A_1D_1A_2D_2A_3D_3A_4C_4$ and straight line l.

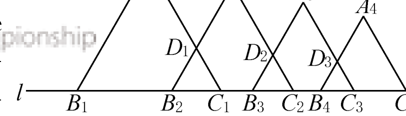

Fig. 11

Relay Round Answers

First Round

1A. $-\frac{5}{3}$.

1B. 25cm^2.

Second Round

2A. 2010.

2B. 350.

Third Round

3A. $\sqrt{3}$.

3B. 376.

Individual Round • Problems

First Round

1. Find the last digit for $2^0+2^1+2^2+\cdots+2^{2012}$.
2. It take x days for A, B, and C to do a job together. If it takes 5 extra days when A works alone and it takes 1 extra day when B works alone and it takes $2x$ extra days when C works alone, find x.
3. The graph of the linear function $y=kx+b$ passes through the point $P(1, 4)$. It also intersects positive x-axis and positive y-axis at A and B, respectively. Let O be the origin. When the area of $\triangle AOB$ is smallest, find the value of k and b.
4. Find all integer values for n so that $n^4-25n^2-70n-49$ is a prime number.

Second Round

5. Given that
$$3x+2y+z=-11,$$
$$x+2y+2z=20,$$
$$x+y+2z=11.$$
Find the average value of x, y, and z.
6. Find the value of $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)}$ when natural number n is getting infinitely large.
7. If $M=(\sqrt{7}+\sqrt{3})^6$, find the largest integer that is not greater than M.
8. The Fig. 1 is a polygon. Find $\angle A_1+\angle A_2+\angle A_3+\cdots+\angle A_{23}+\angle A_{24}$.

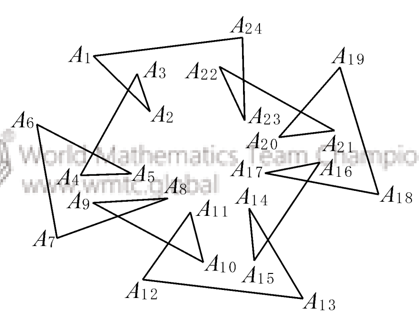

Fig. 1

Third Round

9. Find the number of integers n that will make $\frac{7n+15}{n-3}$ an integer.
10. Given an integer. If subtracted 1996 from it, the result is a perfect square. If subtracted 2008 from it, the result is also a perfect square. Find this integer.
11. If p is a prime number larger than 5, find the remainder when $(p^2+5p+5)^2$ is divided by 120.

12. As in the Fig. 2, use three pieces of colored papers to piece together a large right triangle. The red one is a right triangle with hypotenuse of 29 cm. The blue one is also a right triangle with hypotenuse of 49 cm. The yellow one is a square. Find the sum of the areas of the red and blue triangles.

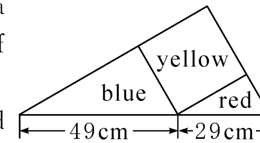

Fig. 2

Fourth Round

13. If $\frac{2xy}{x+y}=-15$, $\frac{yz}{y+z}=6$, and $\frac{7zx}{z+x}=10$ for some real numbers x, y, and z, find $\frac{xyz}{x+y+z}$.

14. Given a number with 28 number of 1's, $\underbrace{111\cdots11}_{28.\ 1's}$, expressed in base 2. Let the number n be its square expressed in base 2. Find the sum of all the digits of n (express the sum in base 10).

Fifth Round

15. How many positive integers not greater than 800 cannot be divided by 2, 3, or 7 evenly?

16. Two squares are inscribed as in the Fig. 3 in the two right triangles $\triangle BAD$ and $\triangle BCD$ in rectangle $ABCD$. If these two squares have areas of $S_1=440$ and $S_2=441$, find the perimeter of rectangle $ABCD$.

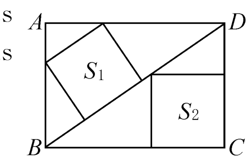

Fig. 3

Individual Round Answers

First Round

1. 1.
2. 1.
3. $k=-4$, $b=8$.
4. -2, -3.

Second Round

5. $\frac{4}{3}$.
6. $\frac{1}{4}$.
7. 7039.
8. 1080°.

Third Round

9. 18.
10. 2012.
11. 1.
12. 710.5cm^2.

Fourth Round

13. $-\frac{15}{2}$.
14. 28.

Fifth Round

15. 229.
16. 924.