2011 **World Mathematics Team Championship**
Advanced Level

Team Round · Problems

1. Suppose $A=\{x\,|\,x^2-x<0\}$ and $B=\left\{x\,\middle|\,\left[x+\frac{1}{3}\right]+\left[x+\frac{2}{3}\right]=1,\ x\in A\right\}$.

Find $A\cap B^C$ where B^C is the complement of B and $[a]$ represents the largest integer not greater than a.

2. Suppose the lengths of the three sides that are opposite to the three interior angles A, B, and C of $\triangle ABC$ are a, b, and c, respectively, and satisfy

$$a^2+b^2+c^2-2\sqrt{7}a-4b-6c+20=0.$$

Find the area of $\triangle ABC$.

3. Suppose $\max\{|a+b|, |a-b|, |2012-a|\}\geqslant C$, where C is a constant, holds true for any real numbers a and b. Find the largest value of C. (*Note*: *max*$\{x, y, z\}$ *represents the largest of* x, y, z)

4. Let $A(-3, 2)$, $B(5, 6)$ and $C(9, -2)$ be three points on the plane. If $ABCD$ is a square, find the coordinates for D and find the area of the part of the square that is in Quadrant II.

5. If the inequality $3^{t^2-2t-1}\geqslant\left(\frac{1}{3}\right)^{2\sqrt{2}\sin^2 x-2\sqrt{2}\sin x\cos x-\sqrt{2}}$ holds for any real number x, find the range of values for t.

6. Find the solution set for inequality $\sqrt{2^x-8}+2\sqrt{4-2^{x-2}}\leqslant\sqrt{2}+\pi$.

7. Let S_n and T_n be the sum of the first n terms of arithmetic sequences $\{a_n\}$ and $\{b_n\}$, respectively. Suppose $\frac{a_5}{b_3+b_{2n-3}}+\frac{a_{2n-5}}{b_7+b_{2n-7}}=\frac{n}{2n+1}$ for any integer n. Find $\frac{S_{23}}{T_{23}}$.

8. If a, b, and c are positive real numbers and $a+b+c=1$, find the integer portion of the number $\sqrt{3a^2+1}+\sqrt{3b^2+1}+\sqrt{3c^2+1}$.

9. As in Fig. 1, there are 12 points A_i $(i=1, 2, 3, \cdots, 12)$ on the ellipse $E: \frac{x^2}{6}+\frac{y^2}{3}=1$. If O is the origin and the included angles between $\overrightarrow{OA_i}$ and $\overrightarrow{OA_{i+1}}$ $(i<12)$ are all equal to $\frac{\pi}{6}$, find $\sum_{i=1}^{12}\frac{1}{|\overrightarrow{OA_i}|^2}$.

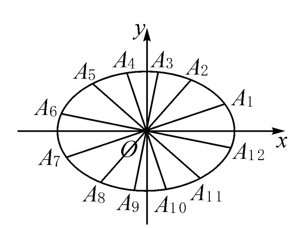

Fig. 1

10. Mr. Wong has \$14 million to invest in two Plans A and B. If he invests in Plan A, the government would subsidize \$2 million. According to estimate, the proportion of profit from Plan A to the square root of the sum of investment and subsidy amount is $\frac{1}{2}$, the proportion of profit from Plan B to the square root of investment is $\frac{\sqrt{3}}{2}$. What is Mr. Wong's maximum total profit in million from these two investments?

11. Let O be the center of the circumscribed sphere of rectangular box $ABCD\text{-}A_1B_1C_1D_1$ with volume $\dfrac{32\pi}{3}$. Label $AB=a$, $BC=b$, and $CC_1=c$.

If $\dfrac{9}{4}$ is the smallest value for $\dfrac{1}{a^2}+\dfrac{4}{b^2}$, find the minimum distance between A and C along the surface of the sphere.

12. Suppose $\theta\in\left(0,\ \dfrac{\pi}{2}\right)$ and equation $x^2-\dfrac{2}{\sin 2\theta}x+1=0$ has a real root $\sin\theta$.

Then what is the value for $\cos\theta$?

13. Suppose equation $x^2-ax+b=0$ has two positive real roots. Then what is the value range for $a+\dfrac{1-b}{a}$?

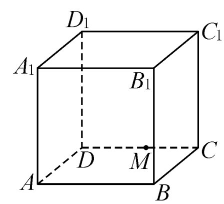

Fig. 2

14. As it is shown in Fig. 2, the unit cube $ABCD\text{-}A_1B_1C_1D_1$ has edge of 1. Let M be a point on edge CD and Ω be a plane section that is inside the cube and passes through points M, A, and C_1. What is this plane section ′s area when this plane forms a dihedral angle (or the angle between these two planes) of $60°$ with plane $ABCD$?

15. Suppose $M=(\sqrt{11}+3)^{2011}$ and (M) represents the decimal portion of M. Find the value of $M\cdot(M)$.

16. Suppose $\angle xOy=2\alpha$, $0°<\alpha<90°$, Oz bisects $\angle xOy$, point A is on Oz and $OA=a$, and MAN is a line that passes through A and intersects Ox and Oy at M and N, respectively.

Find the value for $\dfrac{1}{OM}+\dfrac{1}{ON}$ in terms of α and a.

17. Suppose the area of $\triangle ABC$ is $6\sqrt{3}$, $AB=6$, A is an obtuse angle, D is a point on BC so that $BD=2DC$. If $\overrightarrow{AB}\cdot\overrightarrow{AD}=4$, find $\dfrac{\sin B}{\cos C}$.

18. Suppose F_1 and F_2 are the two foci of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$.

Also, as in the Fig. 3, suppose the right directrix of the ellipse tangent to the circle with center at I and passes through both F_1 and F_2 and that

$$\overrightarrow{IF_1}\cdot\overrightarrow{IF_2}=\frac{a^4}{2a^2-2b^2}.$$

Find the eccentricity of the ellipse.

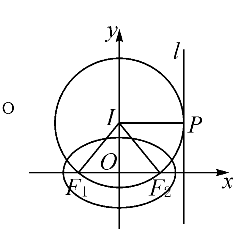

Fig. 3

19. Let $f_0(x)=\dfrac{1}{1-x}$ and $f_1(x)=\dfrac{x-1}{x}$.

Define $f_{n+2}(x)=f_{n+1}(f_n(x))$, $n=0, 1, 2, \cdots$.

Express $f_{2011}(2011)$ in fraction $\dfrac{p}{q}$ of most reduced term. Find $p+q$.

20. If positive numbers x, y, and z satisfy $x+y+z=1$ and $xy+yz+zx+\lambda\sqrt{xyz}\leqslant 1$, what is the largest value for λ.

Team Round Answers

1. $\left\{x \,\middle|\, 0<x<\frac{1}{3} \text{ or } \frac{2}{3}\leqslant x<1\right\}$.
2. $\frac{3}{2}\sqrt{3}$.
3. 1006.
4. $(1, -6)$; $\frac{29}{4}$.
5. $t\leqslant -1$ or $t\geqslant 3$.
6. $[3, 4]$.
7. $\frac{12}{25}$.
8. 3.
9. 3.
10. 4.
11. $\frac{2\pi}{3}$.
12. $\frac{\sqrt{5}-1}{2}$.
13. $[\sqrt{3}, +\infty)$.
14. $\sqrt{5}-1$.
15. 2^{2011}.
16. $\frac{2\cos\alpha}{a}$.
17. $\frac{2\sqrt{3}}{7}$.
18. $\frac{\sqrt{2}}{2}$.
19. 4021.
20. $2\sqrt{3}$.

Relay Round · Problems

First Round

1A. There are three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ on a plane as in the Fig. 1. Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OA}$, $\overrightarrow{OC}$ have included angles of 150° and 60°, respectively. Also, $|\overrightarrow{OA}|=|\overrightarrow{OB}|=1$ and $|\overrightarrow{OC}|=2$. If $\overrightarrow{OC}=\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}$ where λ, $\mu\in R$, find $\lambda^2+\mu^2$.

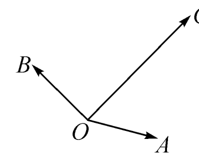

Fig. 1

1B. Let T = TNYWR (The Number You Will Receive).
The real value function f is defined as $f(x+y)=f(x)+y$ for any real numbers x and y.
If $f(T)=2011$, find $f(2011)$.

Second Round

2A. Find the sum of all real roots of the equation $(x+1)(x^2+1)(x^3+1)=30x^3$.

2B. Let T = TNYWR.
Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points moving along the parabola $y^2=4x$ and ellipse $\frac{x^2}{4}+\frac{y^2}{T}=1$, respectively. Let $N(1, 0)$ be a fixed point. If $AB \parallel x$-axis and $x_1<x_2$, find the interval where $\triangle NAB$ can take as its perimeter.

Third Round

3A. Let B be the reflection point of $A(4, 1)$ over the axis of symmetry line $x-y-1=0$.
Find the smallest value for 9^a+27^b+1 where the line $ax+by-2=0$ passes through B.

3B. Let T=TNYWR.
If the function $f(x)=\sqrt{T+2tx}-x$ takes on largest value M where t and M are positive natural numbers, find M.

Relay Round Answers

First Round

1A. 28.

1B. 3994.

Second Round

2A. 3.

2B. $\left(\frac{10}{3}, 4\right)$.

Third Round

3A. 7.

3B. 4.

Individual Round · Problems

First Round

1. Solve the inequality $\sqrt{x+5}>x-1$.
2. If we define a function $f(x+a)=|x-2|-|x+2|$ and $f[f(a)]=3$, find the value for a.
3. If the coefficients a, b, and c of the quadratic equation $ax^2+bx+c=0(abc\neq0)$ form a geometric sequence and the ratio of its two roots x_1 and x_2 is λ, find $\lambda+\dfrac{1}{\lambda}$.
4. Let $P(1, 1)$ be a point inside circle $x^2+y^2=4$ and let AB and CD be two chords of the circle passing through P. If the tangents to A and B intersect at M and the tangents to C and D intersect at N, find the equation of the line MN.

Second Round

5. Suppose a rectangular box has integer edge lengths and its main diagonal is 25. What is the smallest face in terms of area among all such possible boxes?
6. If real numbers x, y, and z satisfy the equation
$$x^2+2y^2+5z^2+2xy+4yz-2x+2y+2z+11=0,$$
find the range of values $x+2y+3z$ takes.
7. Let $M(x_0, y_0)$ be a point inside circle $x^2+y^2=r^2(r>0)$ and $x_0y_0\neq0$. Find the number of points that the line $x_0x+y_0y=r^2$ intersects the circle.
8. As in the Fig. 1, AB_1C_1, $C_1B_2C_2$ and $C_2B_3C_3$ are equilateral triangles all with edge length of 2 sitting on a straight line next to each other with common vertices at C_1 and C_2. Vertices B_1, B_2, B_3 are all on the same side of that line. Suppose there are 10 points P_1, P_2, $\cdots$, P_{10} on side B_3C_3 and define
$$m_i=\overrightarrow{AB_2}\cdot\overrightarrow{AP_i}(i=1, 2, \cdots, 10).$$
Find $m_1+m_2+\cdots+m_{10}$.

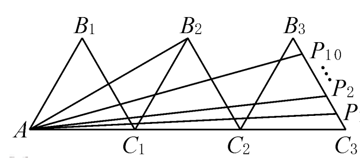

Fig. 1

Third Round

9. If x is an acute angle and $\tan x=\sqrt{2}-1$, find x.
10. In $\triangle ABC$, let a, b, and c be the opposite sides of $\angle A$, $\angle B$, and $\angle C$, respectively. If $\angle C=3\angle B$, compare the size relationship between c and $3b$.
11. Suppose the three real roots for the equation $x^3-(4+d)x^2+5dx-d^2=0$, where d is a natural number, represent the square of some right triangle's three sides. Find d.
12. Suppose F_1 and F_2 are both foci for the ellipse $\dfrac{x^2}{m}+\dfrac{y^2}{n}=1$ and foci for the hyperbola $\dfrac{x^2}{p}-\dfrac{y^2}{q}=1(m, n, p, q\in R^+)$. If M is an intersection of the ellipse and the hyperbola and
$$|\overrightarrow{MF_1}|\cdot|\overrightarrow{MF_2}|=1.$$
Find the minimum value for $\dfrac{1}{2n}+\dfrac{2}{q}$.

Fourth Round

13. If $x=\sin 35^\circ\cos 65^\circ-\cos 65^\circ\cos 5^\circ-\cos 55^\circ\cos 5^\circ$, what is the value for x?

14. A-BCD is a regular tetrahedron with edge length of 24. O is a sphere inscribed inside the tetrahedron. O_1 is a small sphere that is tangent to the upper three sides of the tetrahedron and the large sphere. What is the volume of that small sphere? (*Express your answer in terms of* π)

Fifth Round

15. Find the intervals for x where the function $y=\left|\log_2|x+1|\right|$ is monotonically decreasing.

16. Determine S_{100} if $S_{n+1}=\dfrac{S_n}{1+nS_n}$, $n=0, 1, 2, 3, \cdots$ and $S_0=\dfrac{1}{100}$. (Express the answer in fraction of lowest term)

Individual Round Answers

First Round

1. $[-5, 4)$.
2. $\dfrac{3}{2}$.
3. -1.
4. $x+y=4$.

Second Round

5. 108.
6. $[-1, 5]$.
7. 0.
8. 180.

Third Round

9. 22.5°.
10. $c<3b$.
11. 4.
12. $\dfrac{9}{2}$.

Fourth Round

13. $-\dfrac{3}{4}$.
14. $8\sqrt{6}\pi$.

Fifth Round

15. $(-\infty, -2]$ and $(-1, 0]$.
16. $\dfrac{1}{5050}$.