2010 **World Mathematics Team Championship**
Advanced Level
Team Round · Problems

1. Given two non-empty sets A and B. In the Venn's diagram shown in Fig. 1, define $A※B$ to be the shaded area. If

$$M=\{x\mid y=\sqrt{-x^2+3x+10}\},\text{ and } N=\{y\mid y=3^x-1\},$$

then $M※N=$________.

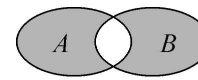

Fig. 1

2. Given functions $f(x)=x^2-3$ and $g(x)=m(x-1)$. If for any $x_0\in[-3,3]$ there exists $x'\in[-3,3]$ such that $g(x')=f(x_0)$, then the value range for the real number m is ______.

3. If the solution set of x for the inequality $mx>n$ is $(-\infty,3)$, then the solution set of x for the inequality $(m-n)x+m+n>0$ is ________.

4. Use $[x]$ to represent the largest integer that is not larger than x. If a real number r satisfies $\left[r+\frac{1}{10}\right]+\left[r+\frac{2}{10}\right]+\cdots+\left[r+\frac{9}{10}\right]=122$, then the value of $[10r]$ is ________.

5. Given a quadratic equation $x^2-x\sin\theta+\sin\theta-5=0$ in terms of x. Then this equation's largest root is ________ and its smallest root is ________.

6. If the three straight lines $2x-y+1=0$, $x+y+2=0$, $x+ay=0$ cannot form a triangle, then a must take on values of ________.

7. Suppose that x, y and z are real numbers that satisfy $x+2y+3z=1$ and $yz+zx+xy=-1$, then the value range for $x+y+z$ is ________.

8. Suppose a, b, $c\in\mathbf{R}^+$, $a+b+c=1$ and $M=\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1}$, then the integer part of M is ________.

9. Given a sequence $\{a_n\}$ where $a_1=3$, $a_2=5$ and $a_{n+2}a_n^2=a_{n+1}^3$. Then the formula for $a_n=$______.

10. The root of the equation $\log_m(\sqrt{x^2+1}+x)+\log_m(\sqrt{x^2+2}+x)=\frac{1}{2}\log_m 2$ ($m>0$ and $m\neq1$) is $x=$________.

11. Given $\boldsymbol{a}=(m+2,n)$, $\boldsymbol{b}=(m-2,n-4)$, $\boldsymbol{a}\perp\boldsymbol{b}$ and $|\boldsymbol{a}|+|\boldsymbol{b}|=8$, then $m+n=$______.

12. The area of $\triangle ABC$ is 2. Let AD, BE and CF be the three medians intersecting at point G, and points H, I and J be on these three medians, respectively, such that $AH:HD=1:1$, $BI:IE=1:2$, and $CJ:JF=1:3$, then the area of $\triangle HIJ$ is ________.

13. Suppose $a+b+c=12$ and $ab+bc+ca=45$ where a, b and c are positive real numbers, then the maximum possible value of abc is ________.

14. Suppose $C: x^2+(y-1)^2=r^2$ and $y=\sin x$ have only one intersection and the x-coordinate of this intersection point is α, then $\dfrac{\sin\alpha+\sin3\alpha-4\cos^2\alpha}{\alpha\cdot\cos\alpha}=$________.

15. The area of the region on the xy- plane over which P ranges,

$P\in\left\{(x,y)\left|\dfrac{(x-\cos\theta)^2}{4}+\left(y-\dfrac{1}{2}\sin\theta\right)^2=1,\ \theta\in\mathbf{R}\right.\right\}$, is ________.

16. Suppose $[r(\cos\theta+\mathrm{i}\sin\theta)]^{10}=\dfrac{243(1-\sqrt{3})\mathrm{i}-243(1+\sqrt{3})}{64(1+\mathrm{i})}$. If $r>0$, $0<\theta<\dfrac{\pi}{3}$, then $r=$________, $\theta=$________.

17. From a point $(\sqrt{5},\sqrt{2})$ inside the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$, draw two chords AB and CD with A, B, C and D on the ellipse. From A and B, draw two tangents so that they intersect at E. From C and D draw another two tangents so that they intersect at F. Then the linear equation for the line that connects E and F is ________.

18. For each $n=1, 2, 3, \cdots$, straight line $y=x+n+1$ intersects the parabola $x^2=\dfrac{1}{8}\left(y-\dfrac{1}{32}\right)$ at two points. Let $|A_nB_n|$ denote the length of the chord connecting the intersecting 2 points for each n. Define $a_n=\dfrac{1}{n|A_nB_n|^2}$. Let S_n be the sum of the first n terms in the sequence $\{a_n\}$. Then $S_{2010}=$________.

19. Consider a cube with edge length a that is hanging above a plane α. Parallel light rays that are perpendicular to plane α project this cube onto α to form a shadow region and then circles are drawn inside that projection region. Then the diameter of the largest circle is ________.

20. Let the radius of $\odot A$ be 2 and point P is outside of $\odot A$. Also, let straight line PM tangent circle $\odot A$ at point M and that $\cos\angle MPA=\dfrac{\sqrt{3}}{2}$. Now if we use the line segment PA as the axis and rotate the line segment PM and $\odot A$ around PA for one revolution. Then the volume of the part of the solid formed by rotating PA that is outside of the sphere formed by $\odot A$ is ________.

Team Round Answers

1. $[-2,-1]\cup(5,+\infty)$.

2. $\left(-\infty,-\dfrac{3}{2}\right]\cup[3,+\infty)$.

3. $(2,+\infty)$.

4. 135.

5. $\dfrac{1+\sqrt{17}}{2}$; -3.

6. ±1 or $-\dfrac{1}{2}$.

7. $\left[\dfrac{3-3\sqrt{3}}{4},\dfrac{3+3\sqrt{3}}{4}\right]$.

8. 4.

9. $3\cdot\left(\dfrac{5}{3}\right)^{2^{n-1}-1}$.

10. 0.

11. 2.

12. $\dfrac{19}{48}$.

13. 54.

14. -4.

15. 4π.

16. $\dfrac{\sqrt{6}}{2}$; $\dfrac{\pi}{12}$, $\dfrac{17}{60}\pi$.

17. $\dfrac{\sqrt{5}}{9}x+\dfrac{\sqrt{2}}{5}y=1$.

18. $\dfrac{2010}{2011}$.

19. $\sqrt{2}a$.

20. $\dfrac{4\pi}{3}$.

Relay Round · Problems

First Round

1A. Solve the inequality for a, $2a^2+(4\sqrt{2}-7)a+(3-2\sqrt{2})<0$.

1B. Let $A=$ the answer passed from your teammate. Solve the inequality $2\log_a(x-2)>\log_a 2+\log_a(6-x)$ using value of a from A.

Second Round

2A. Give a sequence $\{a_n\}$ such that $a_1=1$ and $2a_{n+1}+a_n=2^{n+1}$. Find a_n.

2B. Let $A=$ the answer passed from your teammate. Suppose the hypotenuse of a right triangle has a length of 5 and the cosine of one of its angles is $\lim\limits_{n\to+\infty}\dfrac{A}{2^{n-1}}$. If we use one of sides of this right triangle as axis and rotate the triangle around this axis to get a solid of revolution, find the largest volume of all such solids.

Third Round

3A. Suppose that the directrix of a parabola $y^2=2px\,(p>0)$ is tangent to the circle $x^2+y^2-4x+2y-4=0$. Find p.

3B. Let $A=$ the answer passed from your teammate. A bag has x black and $(15-x)$ white balls that are identical in every way except for their colors. Suppose that one takes out $(A+1)$ balls randomly. Use $P(x)$ to represent the probability of getting A black balls and 1 white ball. Then $P(x)$ has the largest value when $x=$________.

Relay Round Answers

First Round

1A. $\left(3-2\sqrt{2},\dfrac{1}{2}\right)$.

1B. $(2,4)$.

Second Round

2A. $\dfrac{2}{5}[2^n-(-1)^n\cdot 2^{-n}]$.

2B. 16π.

Third Round

3A. 2.

3B. 10.

Individual Round • Problems

First Round

1. If $f(x)=\dfrac{1}{1+2^{\lg x}}+\dfrac{1}{1+4^{\lg x}}+\dfrac{1}{1+8^{\lg x}}$, then $f(x)+f\left(\dfrac{1}{x}\right)=$ ________.

2. When $x\to+\infty$, the graph of function $f(x)=\dfrac{x(3^x-5)}{3^x+1}$ is approaching the graph of which of the following function ? Answer: ________.

(A) $y=x$. (B) $y=\dfrac{x}{3}$. (C) $y=\dfrac{1}{x}$. (D) $y=\dfrac{x}{2}$.

3. The solution to $2^{|x|}-2|x|=22$ is $x=$ ________.

4. The number of non-empty subsets of set $A=\left\{x\,\middle|\,\dfrac{x^2-30}{|x|-2}<0,\ x\in\mathbf{Z}\right\}$ is ________ where $\mathbf{Z}$ is the set of integers.

Second Round

5. If $M=7\times10^{753}+2\times10^{573}+10^{372}+5\times10^{357}+4\times10^{2}+2\times10$, then the sum of the elements in $\{3,4,5,6,8,9,10,11\}$ that are factors of M is ________.

6. Suppose that real numbers x and y satisfy $x^2+y^2=2$, and $x+\sqrt{3}y\geqslant\sqrt{6}$. Then the maximum value of $x+y$ is ________.

7. If a sequence $\{a_n\}$ is defined as $a_1=2$, $a_2=5$ and $a_{n+1}=a_n+a_{n+2}$, then $a_{2010}=$ ________.

8. If $\{x\mid-1\leqslant x\leqslant3\}$ is the solution set for inequality $\sqrt{-x^2+2x+3}-a(x-4)>0$, then the value range for a is ________.

Third Round

9. If x and y are positive real numbers that satisfy $(\sqrt{1+x^2}-x+1)(\sqrt{1+y^2}-y+1)=2$, then $xy=$ ________.

10. The edges AA_1, AB and AD in a parallelepiped $ABCD\text{-}A_1B_1C_1D_1$ have lengths of 2, 3 and 4, respectively. If both of their angles are 60°, then the length of this parallelepiped's diagonal AC_1 is ________.

11. Define a sequence $\{a_n\}$ to be $a_1=2$, $a_n=\dfrac{4}{5}a_{n-1}$ when $n\geqslant2$. Starting from point M, point M_n moves right by a_1 units arriving at point M_1. Then the point makes a left turn for 90° and moves forward by a_2 units arriving at point M_2. Make another 90° left turn and move forward by a_3 units arriving at point M_3. Keeping this process, the point M_n approaches to a fixed point N infinitely. Use M as the coordinate origin and the straight line through M toward right as
the x-axis to form a rectangular Cartesian coordinate, then the coordinate of the point N is ________.

12. Suppose $\angle B=15^\circ$ and $\angle C=30^\circ$ in $\triangle ABC$. Let D be a point on BC so that AD is the angle bisector for $\angle A$. If $\overrightarrow{AD}=\lambda\overrightarrow{AB}+\mu\overrightarrow{AC}(\lambda,\mu\in\mathbf{R})$, then the value of $\dfrac{\lambda}{\mu}$ is ________.

Fourth Round

13. For any real number θ, move the straight line $l: x\cos\theta + y\sin\theta = 2$ and form a region. The area of the region is ________.

14. Given a rectangle $ABCD$ with $AB=4$ and $BC=6$ and a square $AEFG$ of length $\sqrt{13}$ that shares a vertex A with the rectangle $ABCD$. Suppose this square, on the same plane as $ABCD$, rotates around point A for one revolution, then the value range for the length CE is ________.

Fifth Round

15. As in Fig. 1 on the right, the edge of this cube is a, point M is on CD so that $CM = \frac{a}{4}$, and point N is on GH so that $GN = \frac{2a}{3}$. Suppose a plane that passes through points B, M and N and divides this cube into upper and lower parts, then the volume of the lower part is ________.

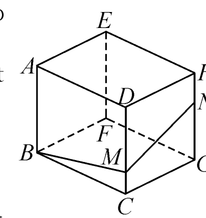

Fig. 1

16. Given the following conditions: straight lines $l_1: x+3y=2$, $l_2: y=kx\ (k>0)$, and ellipse $C: x^2+4y^2=4$. l_2 intersects C at points M and N where M is in Quadrant Ⅲ. Suppose that l_1 intersects l_2 at point P, O is the origin, and $|MO|$, $|OP|$ and $|PN|$ form an arithmetic sequence. Then $k=$ ________.

Individual Round Answers

First Round

1. 3.

2. (A).

3. ± 5.

4. 63.

Second Round

5. 39.

6. 2.

7. -3.

8. $(0, +\infty)$.

Third Round

9. 1.

10. $\sqrt{55}$.

11. $\left(\frac{50}{41}, \frac{40}{41}\right)$.

12. $\frac{\sqrt{6}-\sqrt{2}}{2}$.

Fourth Round

13. 4π.

14. $[\sqrt{13}, 3\sqrt{13}]$.

Fifth Round

15. $\frac{1}{3}a^3$.

16. $\frac{5}{24}$.